Discrete Structuren)
Hertentamen, 27-06-2007

The problems are to be solved within 3 hrs.

The use of supporting material (books, notes, calculators) is not allowed.

In each problem you can obtain 10 points, i.e. 100 in total. Your partial result for the first 5 problems may be replaced by your grade in the midterm exam ($\times 5$), provided the grade was ≥ 5.5.

Some useful hints:

- Really read these hints.
- Give precise arguments for all your answers.
- You can write in English or Dutch, but in any case use a **readable font!**
- Counterexamples prove that a statement is not true, but positive examples do not prove general validity.
- If you refer to the hand-out sheet, numbers of implications etc. are sufficient.

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1. Prove that the following proposition

$$(\neg p \to r) \leftrightarrow ((r \to q) \to (p \vee r))$$

is a tautology. Use the form of an *annotated linear proof* (geannoteerd lineair bewijs).

2. Prove (by cases) that $|x + y| \leq |x| + |y|$ for $x, y \in \mathbb{R}$.

3. Prove by (infinite) mathematical induction: $\sum_{i=0}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$ for $n \in \mathbb{N}$

4. Give an explicit expression for the sequence s_n, defined by

$$\begin{aligned} s_0 &= 1 \\ s_1 &= 1 \\ s_n &= 2s_{n-1} + 2s_{n-2} \quad \text{for } n \geq 2 \end{aligned}$$

5.
(a) Let $s(n)$ $(n \in \mathbb{N})$ be a sequence. Define the meaning of $s(n) = O(n)$ and of $s(n) = \Theta(n)$.
(b) Are the following statemens true or false? (Give precise arguments!)

$$2^{2n} = O(2^n) \qquad 2^{n+1} = \Theta(2^n)$$

6. Let the relation $\sim$ on $\mathbb{N}$ be defined by: $m \sim n$ if and only if $5 \mid (m-n)$ (i.e. 5 divides $m-n$). Show explicitly that $\sim$ satisfies the properties of an equivalence relation. What are the equivalence classes of $\sim$?

7.

(a) Show that the proposition

$$[\exists x\, p(x)] \wedge [\exists x\, q(x)] \rightarrow \exists x\, [p(x) \wedge q(x)]$$

is <u>not</u> a tautology. You can do this by giving examples for $p(x)$ and $q(x)$ for which the proposition is false.

(b) Show that the proposition

$$\exists x \forall y\; p(x,y) \rightarrow \forall x \exists y\; p(x,y)$$

is <u>not</u> a tautology. Again it is sufficient to show that the proposition is false for a particular $p(x,y)$.

8. Let $\boldsymbol{A}$ be the Boolean matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Calculate $\boldsymbol{A} * \boldsymbol{A}$.

(b) Is the relation corresponding to $\boldsymbol{A}$ transitive? Explain your answer!

(c) Which matrices represent the symmetric closure, the reflective closure, and the transitive closure of the relation corresponding to $\boldsymbol{A}$?

9. After having graduated you have been hired by a manufacturer of computer hardware. Your first task is to specify a scheme for the serial number of a new product. You decide on using alphanumerical characters, i.e. the 26 capital letters and the 10 digits.
Your company does not expect to manufacture more then 1000000000 (i.e. 10^9) of these devices. Out of how many alphanumerical characters should the serial number consist, such that there will be a unique serial number for each manufactured device and the serial number of each device is as short as possible?
You should not give the result as a number; it is sufficient to provide an analytic expression, e.g. exp[12] instead of 162754.7914....

10.

(a) How many edges are there in a complete graph with $n = 11$ vertices?

(b) How many edges are there in a binary rooted tree with $n = 23$ vertices?

(c) How many edges are there in a ternary rooted tree with $n = 23$ vertices?